

Q-States Potts model on a random planar lattice

Jean-Marc DAUL

Laboratoire de Physique Théorique de l'Ecole Normale Supérieure¹,
24 rue Lhomond, 75231 Paris CEDEX 05, France.

Abstract

We propose a matrix-model derivation of the scaling exponents of the critical and tricritical q-states Potts model coupled to gravity on a sphere. In close analogy with the $O(n)$ model, we reduce the determination of the one-loop-to-vacuum expectation to the resolution of algebraic equations; and find the explicit scaling law for the case $q=3$.

LPTENS 94/xx

November 1994

¹Unité Propre du Centre National de la Recherche Scientifique, associée à l'École Normale Supérieure et à l'Université de Paris-Sud.

e-mail address : daul@physique.ens.fr

1 Matrix realization of the Potts model

Matrix models have proved to be an efficient means to deal with the statistical properties of matter coupled to two-dimensional gravity: instead of performing functional averages on the metric and embedding fields, we consider a matrix-valued field theory whose diagrammatics corresponds to the enumeration of simplicial complexes (discretized surfaces) dressed with other (i.e. non-geometrical) degrees of freedom (e.g. Ising, Potts spins, a scalar field...).

Here, we shall examine [1]:

$$Z = \int d\phi \exp -N\text{Tr} \left[\sum_i \frac{m^2 - 1}{2} \phi_i^2 - \sum_{\langle ij \rangle} \phi_i \phi_j + \sum_i \frac{\bar{g}}{3} \phi_i^3 \right] \quad (1)$$

with $\phi_{1\dots q}$ q $N \times N$ hermitean matrices, and $\langle ij \rangle$ denoting any pair of different indices. Expanding in powers of g and performing the gaussian integral, we obtain various ϕ^3 fatgraphs (we represent any propagator with a double-line and corresponding row/column indices attached, and any vertex with a fat Y with the index structure of the trace of a third power $\phi_{ab}\phi_{bc}\phi_{ca}$) where each vertex has been attached a number $i = 1, \dots, q$ (which matrix has been chosen to produce that vertex?). We can as well consider dual graphs: various gluings of colored triangles in an orientable surface without boundary. As for the statistical weight, we get a contribution from the vertices, recognized as a cosmological constant (energy \sim area = number of triangles); and from the propagators, e.g.:

$$\frac{\langle \phi_1 \phi_1 \rangle}{\langle \phi_1 \phi_2 \rangle} = m^2 - q + 1 \quad (2)$$

where the energy of a satisfied link (neighbouring triangles with the same color) re-normalizes the cosmological constant (there is 1.5 link per unit area) and the relative energy of frustrated bonds produces the required statistical weight of the Potts model; the temperature is truly positive, provided that $m^2 > q$. For any m , the average size of the surfaces will grow with \bar{g} , and there is a critical value $\bar{g}_c(m)$ near which the thermodynamic regime is achieved; if we also tune m , the temperature, the spin-correlation length will also diverge, and we shall obtain the critical regime.

So far, diagrams of all genera appear; but they are weighted by $N^{\chi=\text{Euler characteristic}}$ as is easily seen: a factor of N accompanies any vertex; $1/N$ for each propagator; and a choice of a free index ($a = 1, \dots, N$) for every loop in the ϕ^3 fatgraph. Taking the logarithm, we obtain connected diagrams only, and sphere-like diagrams are extracted in the $N \rightarrow \infty$ limit.

In all solved matrix models, a tremendous simplification appears in that limit: variables are changed from ϕ to Ω, λ according to $\phi = \Omega^+ \text{diag}_{1\dots N}(\lambda) \Omega$ and when the integral over the diagonalizing unitary matrices is preformed, one is left with an integral over the N eigenvalues per original matrix with an effective energy $\mathcal{O}(N\text{Tr}) = \mathcal{O}(N^2)$, that is: a semi-classical problem (note that the ground state corresponds to eigenvalues of order 1).

Indeed, the integral over Ω is trivial for a one-matrix model where the energy is invariant under conjugation. For terms coupling two matrices

$$\text{Tr}\phi_1\phi_2 = \text{Tr}\Omega_1^+ \text{diag}(\lambda_1)\Omega_1\Omega_2^+ \text{diag}(\lambda_2)\Omega_2$$

the required integral involves the relative angle $\Omega_1\Omega_2^+$ and is exactly known [3]. However, when the model couples 3 or more matrices in a cyclic way $(\phi_1\phi_2, \phi_2\phi_3, \phi_3\phi_1)$ the relative angles are no more independent, the semi-classical reduction is no longer possible and we do not know how to solve such models - that would correspond to matter with $c > 1$ (e.g. a 2-dimensionnal lattice of matrices to produce surfaces embedded in 2 dimensions).

This problem is only apparent in the case under consideration: the Potts model on a flat regular lattice is known to have central charge less than one, and the cyclic structure of (1) can be disentangled by introducing an auxiliary gaussian matrix symmetrically coupled to the Potts matrices [1, 4]

$$Z = \int dX d\phi \exp -N\text{Tr} \left[\sum_i V(\phi_i) + X^2/2 - \sum_i \phi_i X \right] \quad (3)$$

where $V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\bar{g}}{3}\phi^3$.

Now the relative angles Ω_{X/ϕ_i} are independent and can be integrated out: to solve the problem for planar random graphs, we just have to determine the distribution of the eigenvalues of the auxiliary and Potts matrices in the lowest-energy configuration. We shall note ρ the density of eigenvalues for X , that is $\rho(x) dx =$ the fraction of eigenvalues in $[x, x + dx]$; σ will correspond to the Potts matrices (we assume no symmetry breaking, so that all q matrices have the same spectrum at equilibrium). The associated resolvents are f and g :

$$f(z) = \int \frac{\rho(x)dx}{z - x} \quad (4)$$

and correspondingly for g ; they are holomorphic functions on the complex plane cut along the supports of their respective densities (finite intervals). The densities do depend on m and g , and scaling laws exist between the critical exponents attached to them and critical indices of the spin system: for instance, the string susceptibility γ_{str} which governs the area-dependence of the two-point function is known through the exponent with which the density of eigenvalues vanishes near the edge of its support *exactly* at the critical point $\sigma(y) \sim y^\delta$

$$\gamma_{str} = -(\delta - 1) \quad (5)$$

We shall find the densities in the critical regime.

When we change from hermitean to polar variables, a Jacobian $\Delta(\lambda)^2$ appears: the square of the Vandermonde determinant of the eigenvalues. This leads to (repulsive) interactions between the eigenvalues, to which we shall add the effect of the potential dressed by the interactions between matrices: if we set

$$I(X) = \int d\phi e^{-N\text{Tr}[V(\phi) - X \cdot \phi]} = I(x_1, \dots, x_N) \quad (6)$$

the dressed potential seen by the x 's is $N \sum x_i^2 - q \log(I(x))$ and the classical equilibrium equation reads

$$\frac{2}{N} \sum_{j:j \neq i} \frac{1}{x_i - x_j} - x_i + q w(x_i) = 0 \quad (7)$$

where $w(x_i) = \frac{1}{N} \frac{\partial}{\partial x_i} \log(I) = \langle \phi_{ii} \rangle$, the average being taken for a matrix ϕ that fluctuates in its own cubic potential under the influence of a *diagonal* matrix X with spectrum ρ .

So, we have

$$2\text{Re } f(x) - x + q w(x) = 0, \quad x \in \text{supp } \rho \quad (8)$$

and if we introduce

$$J(\phi) = \int dX e^{-N \text{Tr}[\frac{1}{2} X^2 - X \cdot \phi]} I(X)^{q-1} \quad (9)$$

to express the effect on ϕ of the fluctuations of the auxiliary matrix (that feels the influence of the $q - 1$ other Potts matrices); writing

$$\zeta(\lambda_i) = \frac{1}{N} \frac{\partial}{\partial \lambda_i} \log(J) = \langle X_{ii} \rangle \quad (10)$$

we obtain

$$2\text{Re } g(y) - V'(y) + \zeta(y) = 0, \quad y \in \text{supp } \sigma \quad (11)$$

This Hartree-Fock formulation of the large N situation is similar to that used in [5] to find the common distribution of eigenvalues of matrices connected along a Bethe tree; here, the tree is finite (with $q + 1$ sites), and the density is site-dependent. The resolution will proceed along similar lines, using results of [6].

2 The critical point

We can extract useful information without solving explicitly (8,11), just by using analytic properties of generalized resolvents. Let us introduce

$$W_k(z) = \frac{1}{I} \left(\frac{1}{z - \frac{1}{N} \partial_X} \right)_{kk} \cdot I = \left\langle \left(\frac{1}{z - \phi} \right)_{kk} \right\rangle \quad (12)$$

which we shall also write $W_x(z)$, x being the k -th eigenvalue of X , to which ϕ_{kk} is coupled. We have [6]

$$\frac{1}{N} \sum_j \frac{W_k(z) - W_j(z)}{x_k - x_j} = -\frac{1}{N} \left(\frac{d_X I}{I} \cdot \left\langle \frac{1}{z - \phi} \right\rangle \right)_{kk} + \left\langle \frac{\phi}{z - \phi} \right\rangle_{kk} \quad (13)$$

This formula can be obtained by considering the contracted differential $d_{ab} f_{bc}$ of the matrix-valued function:

$$f : X \mapsto \left\langle \frac{1}{z - \phi} \right\rangle$$

(13) only expresses the equivariance of f

$$f(\Omega X \Omega^+) = \Omega f(X) \Omega^+$$

In the large N limit, (13) reduces to

$$\int dx \rho(x) \frac{W_a(z) - W_x(z)}{a - x} = -w(a)W_a(z) + zW_a(z) - 1 \quad (14)$$

Similarly, we define

$$Z_k(z) = \frac{1}{J} \left(\frac{1}{z - \frac{1}{N} \partial_\phi} \right)_{kk} \cdot J = \left\langle \left(\frac{1}{z - X} \right)_{kk} \right\rangle$$

and we introduce the generalized resolvents

$$\begin{aligned} F(z, s) &= 1 - \int dx \frac{\rho(x)W_x(z)}{s - x} \\ G(z, s) &= 1 - \int dy \frac{\sigma(y)Z_y(z)}{s - y} \end{aligned} \quad (15)$$

Expliciting F we have

$$\begin{aligned} F(z, s) &= 1 - \frac{1}{N} \sum_k \frac{1}{s - x_k} \left\langle \left(\frac{1}{z - \phi} \right)_{kk} \right\rangle \\ &= 1 - \frac{1}{N} \left\langle \text{Tr} \frac{1}{s - X} \frac{1}{z - \phi} \right\rangle \end{aligned} \quad (16)$$

and, as X and ϕ change roles from F to G , $G(s, z) = F(z, s)$.

For large z , $W(z)$ is well defined and $F(z, \cdot)$ has a cut along $\text{supp } \rho$:

$$\begin{aligned} F(z, a \pm i\epsilon) &= 1 - \oint dx \frac{\rho(x)W_x(z)}{a - x} \pm i\pi \rho(a)W_a(z) \\ &= W_a(z)[z - w(a) - \text{Re } f(a) \pm i\pi \rho(a)] \end{aligned} \quad (17)$$

in virtue of (14). We assume that w and ζ can be analytically continued near the supports of ρ, σ (independently below and above the cuts: we shall see that the edges are branching points) and we set

$$u = w + f, \quad v = \zeta + g \quad (18)$$

so that

$$\begin{aligned} F(z, a) &= W_a(z)(z - u(a)) \\ G(z, b) &= Z_b(z)(z - v(b)) \end{aligned} \quad (19)$$

and, as these functions go to one at infinity (16), we can write a dispersion integral for their phases

$$\begin{aligned} F(z, s) &= \exp - \oint_{\text{supp } \rho} \frac{d\tau}{2i\pi} \frac{1}{\tau - s} \log [z - u(\tau)] \\ G(z, s) &= \exp - \oint_{\text{supp } \sigma} \frac{d\lambda}{2i\pi} \frac{1}{\lambda - s} \log [z - v(\lambda)] \end{aligned} \quad (20)$$

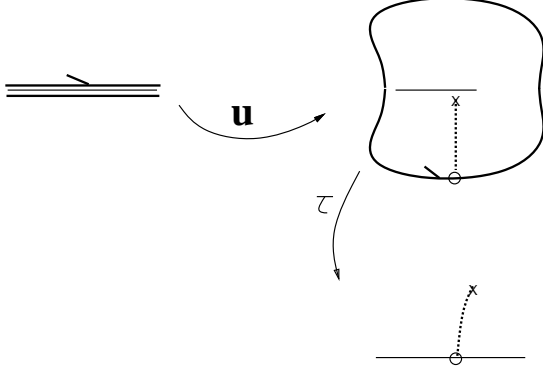


Figure 1: In $u \circ \tau = id$, we evaluate u on the sheet on which lies the value taken by τ after analytic continuation (dashed line).

with contour integrals going counterclockwise along $\text{supp } \pm i\epsilon$, this being a valid representation for large s, z . (Proof: for fixed large z , the ratio of both sides is a holomorphic function of s , with no cut, going to 1 as $s \rightarrow \infty$, thus equal to 1.) We note $[\alpha, \beta] = \text{supp } \rho$.

Let us express the symmetry property: $G(s, z) = F(z, s)$: we change the variable in the first contour integral from τ to $u(\tau)$ and then integrate by parts to obtain

$$F(z, s) = \exp \oint_{u(C)} \frac{du}{2i\pi} \frac{\log [\tau(u) - s]}{u - z} \quad (21)$$

To be precise, the former integral involved u_{\pm} along $[\alpha, \beta] \pm i\epsilon$; the new expression involves the integrals along two arcs, joining $u_+(\alpha) = u_-(\alpha)$ (for $\rho(\alpha) = 0$) to $u_+(\beta) = u_-(\beta)$, of the corresponding inverse function τ : see Figure 1.

We note that the upper segment is mapped onto the lower arc because $\text{Im } u_+ < 0$: so, the orientation of the image contour depends on the order of $u(\alpha), u(\beta)$.

We expect as few singularities as possible for u and τ , because we introduced the smallest possible number of coupling constants (should we have taken more, we could have produced higher order criticality through a richer singularity structure): we assume we can flatten the integration contour to $(u(\alpha), u(\beta))$ and analytically continue τ without encountering any singularity, so that

$$\begin{aligned} F(z, s) &= \exp \oint_{(u(\alpha), u(\beta))} \frac{du}{2i\pi} \frac{1}{u - z} \log [\tau(u) - s] \\ &= G(s, z) = \exp - \oint_{\text{supp } \sigma} \frac{d\lambda}{2i\pi} \frac{1}{\lambda - z} \log [v(\lambda) - s] \end{aligned} \quad (22)$$

We knew that equality for large z, s : under that form, contour integrals are seen to be equal (up to $2i\pi$, but they vanish at infinity!) for large s and any non-real z . Considering the discontinuity on the real z -axis, we obtain for large s :

$$\chi_{\lambda \in (u(\alpha), u(\beta))} \log \left[\frac{\tau_+(\lambda) - s}{\tau_-(\lambda) - s} \right] = \pm \chi_{\lambda \in \text{supp } \sigma} \log \left[\frac{v_+(\lambda) - s}{v_-(\lambda) - s} \right] \quad (23)$$

where the sign depends on the relative order of $u(\alpha), u(\beta)$. We conclude that $(u(\alpha), u(\beta))$ covers $\text{supp } \sigma$ and that

$$\tau_\epsilon(\lambda) = v_{\pm\epsilon}(\lambda), \lambda \in \text{supp } \sigma \quad (24)$$

For instance, if u preserves the order of α, β : $\tau_\pm = v_\pm$ on $\text{supp } \sigma$ and

$$\lambda = u_+[v_-(\lambda)] \quad (25)$$

for $\lambda \in \text{supp } \sigma$ and so, for any λ by analytic continuation. In any case, attention shall be paid to determine the u-sheet on which $v(\lambda)$ has to be evaluated: one has to follow the value of τ as its argument moves from $u(\mathcal{C})$ towards the real axis.

This inversion relation $u \circ v = id$ always holds, in the planar limit, whether we are at the critical point or not. However, if v is singular at λ then u is at $v(\lambda)$: but at the critical point, any function has all its singularities lying at the same place (see section 3), and we expect each density to develop a singularity at one edge of its support; so, these singular edges are exchanged under u, v .

We call γ, δ the exponents with which ρ, σ vanish at the singular edges: they govern the corresponding leading singularities of f, g . For example, if $\rho(\alpha + \epsilon) = cst \epsilon^\gamma + \dots$ we have

$$f(\alpha + \zeta) = \text{regular} + \frac{(cst > 0)}{\sin \pi \gamma} (-\zeta)^\gamma + \dots$$

where the γ -th power of a positive number is taken positive. Using (8,11) we obtain similar expansions for u, v , e.g. (singularities at the left edge)

$$\begin{aligned} u_\pm(\alpha + z) &= \text{reg} + C \sin \pi \gamma \left[\frac{q-2}{q} \cos \pi \gamma \mp i \sin \pi \gamma \right] z^\gamma + \dots \\ v_\pm(u(\alpha) + z) &= \text{reg} + C' \sin \pi \delta \left[-\cos \pi \delta \mp i \sin \pi \delta \right] z^\delta + \dots \end{aligned} \quad (26)$$

with positive constants $C^{(\prime)}$. Moreover, we expect $1 < \delta < 2$ (see [7], [8], and section 3).

For that singularity to disappear in $u \circ v = id$, γ has to be equal to δ or to its inverse; in the latter case, the linear term in the regular part of v necessarily vanishes. In the first case, we write that some linear combination of C, C' vanishes and obtain a relation between the phases of the bracketed terms of (26). In the other case ($\gamma\delta = 1$), the requirement that $z = (\dots)(z^\delta)^\gamma$ also gives a relation between these phases.

Investigating all the possible situations (respective positions of the singular edges, relation between γ and δ) we discover that, for singularities occuring both at the same edge:

$$\begin{aligned} \text{either } \gamma &= \delta^{-1} \quad \text{and} \quad \cos \pi(1 - 1/\delta) = \sqrt{q}/2 \\ \text{or } \gamma &= \delta \quad \text{and} \quad \cos \pi(\delta - 1) = \sqrt{q}/2 \end{aligned}$$

The case when one singularity occurs at the left edge and the other on the right edge, appears to be inconsistent: this is related to our choice of sign of the $X.\phi$ term in (3);

with that choice X is localized near $\sum_i \phi_i$, and we understand that the spectra become simultaneously singular at corresponding edges.

Expressing our results in terms of $m = -1/\gamma_{str}$ (parametrization of $m, m+1$ unitary models) we find

$$\begin{aligned} \gamma &= \delta^{-1} \quad \text{and} \quad \sqrt{q}/2 = \cos \pi/(m+1) \\ \gamma &= \delta \quad \text{and} \quad \sqrt{q}/2 = \cos \pi/m \end{aligned} \tag{27}$$

and recognize the (flat space - regular lattice) critical and tricritical Potts model [9]. In particular, we recover the unitary (5,6) model as the critical 3-states Potts model. This result shows that it is important not to confuse between the exponents of ρ and *sigma*, otherwise one fails to identify the critical Potts model.

Before turning to the study of the vicinity of the critical point, a last remark is in order: we said that the existence of a singularity for u at λ implied a singularity for v at $u(\lambda)$. This is true exactly at the critical point, but not in its neighbourhood: out of the critical point, densities of eigenvalues vanish as square-roots at the ends of their supports (as in the simple one-matrix case [7]; see also the scaling functions in the next section). Then, $u(\alpha + z)$ is an invertible power series in \sqrt{z} , and the condition on the inverse function is a zero of its derivative $v'(u(\alpha)) = 0$. When we approach the critical point, that zero disappears at the place where two cuts merge (consider the example of $\sqrt{\epsilon^2 - x^2}$ as $\epsilon \rightarrow 0$) and the fractional order of the zero of the density at the edge of the support is increased: this will be transparent with the more explicit results of the following section.

3 The scaling regime

For any values of the temperature and the cosmological constant, we can use known results about the external field problem (Kazakov and Kostov, in [4]; see also [11]) to compute exactly w in the large N limit: setting $\alpha = 1/m$, $g = \bar{g}/m^3$,

$$w(x)/\alpha = \frac{\sqrt{\alpha x + c}}{\sqrt{g}} - \frac{1}{2g} + 1/2 \int \frac{\rho(t) dt}{\sqrt{\alpha t + c}} \frac{1}{\sqrt{\alpha x + c} + \sqrt{\alpha t + c}} \tag{28}$$

$w(x)$ being the restriction of a holomorphic function with a semi-infinite cut $(-\infty, -c]$ whose location, at the left of the support of ρ , is given by

$$c + \sqrt{g} \int \frac{\rho(x) dx}{\sqrt{\alpha x + c}} = \frac{1}{4g} \tag{29}$$

(for positive g , which we shall assume is the case; for negative g , the cut is located to the right of the support of ρ)

The eigenvalues of X are in the lowest-energy configuration when

$$x = 2 \oint \frac{dy \rho(y)}{x-y} + q\alpha \left(\frac{\sqrt{\alpha x + c}}{\sqrt{g}} - \frac{1}{2g} + 1/2 \int \frac{\rho(t) dt}{\sqrt{\alpha t + c}} \frac{1}{\sqrt{\alpha x + c} + \sqrt{\alpha t + c}} \right) \quad (30)$$

on the support of ρ . Changing notations to

$$u = \sqrt{\alpha t + c}, \pi(u) = \rho(t) \quad (31)$$

and introducing

$$G(z) = \int \frac{du \pi(u)}{z-u} \quad (32)$$

the equilibrium equation reads

$$2\text{Re } G(z) + (2-q)G(-z) = P(z) \quad (33)$$

with

$$P(z) = \frac{z^2 - c}{\alpha} - \frac{q\alpha}{\sqrt{g}}z + \frac{q\alpha}{2g} \quad (34)$$

Note that π is not normalized, but

$$\frac{1}{4g} = c + \frac{\alpha\sqrt{g}}{2} \int \pi \quad (35)$$

while the normalization of ρ now reads

$$\int u\pi(u) du = \alpha/2 \quad (36)$$

When $q = 2$ this equation is easily solved by a dispersion integral and we recover the known results about the Ising model on a dynamical lattice, with

$$\alpha_c^2 = \frac{\sqrt{7}-1}{12}, g/\alpha^3 = \sqrt{10} \quad (37)$$

($\alpha_c < .5$ effectively corresponds to some positive temperature), and the usual scaling law for the resolvent (one-loop function).

For $q = 0$ or 4 , (33) has a Z_2 symmetry ($z \rightleftharpoons -z$) and its solution is expressed through elliptic integrals; we recall that $q \rightarrow 0$ corresponds to the statistics of tree-like polymers on a random lattice and has been solved in [1].

In the general case, the critical point is reached when the support of π , $[a, b]$, fuses with its mirror image ($a \rightarrow 0$), that is, when the singularity of w reaches the support of ρ . The scaling behaviour of G

$$G(a \cosh \theta) \sim a^{(\dots)} g(\theta) \quad (38)$$

is imposed by (33)

$$g(\theta) + g(\theta + 2i\pi) + (2-q)g(\theta + i\pi) = 0 \quad (39)$$

that is

$$g(\theta) = cst \cosh \kappa \theta \quad (40)$$

with $\cos \kappa \pi = q/2 - 1$ which corresponds to the exponent found in the first section for the two-critical Potts model.

So, we found the scaling behaviour of the resolvent of π ; to obtain the one-loop function for a Potts matrix, we shall compute the resolvent of ρ and use the inversion formula. To do so, however, we need to know the position of the support of ρ which involves c : and this parameter is left undetermined when we extract the singular part of G in the scaling limit. To assert a definite conclusion about ρ (and not π) we need compute c : that we did for $q = 3$, an unsolved model up to now.

A general way to proceed with (33)² is to shift G by the polynomial Q that satisfies $2Q(z) + (2 - q)Q(-z) = P(z)$, to obtain a homogeneous equation for $f = G - Q$. Introducing θ such that $\cos \theta = 1 - q/2$ ($\theta = 2\pi/3$ for $q = 3$), we define

$$\phi_{\pm}(z) = f(z) + e^{\pm i\theta} f(-z) \quad (41)$$

These holomorphic functions have a double cut $[-b, -a]$, $[a, b]$, are related by $\phi_{-}(z) = e^{-i\theta} \phi_{+}(-z)$. Through the right cut, ϕ_{+} is continued into $-\phi_{-}$; and through the left cut, into $-e^{2i\theta} \phi_{-}$ according to (33).

When θ is commensurable to π , we can raise ϕ_{\pm} to some integer power so as to obtain holomorphic functions which realize a double-cover of the sphere (with two cuts). Namely, when $q = 3$, we set $\psi_{\pm} = \phi_{\pm}^3$, so that $\psi_{-}(z) = \psi_{+}(-z)$ and the analytic continuation of ψ_{+} through any cut leads to $-\psi_{-}$. At infinity, ψ behaves as z^6/α^3 . We obtain a meromorphic function on the torus (double cover of the two-slit plane) which can be written

$$\psi_{\pm} = \alpha^{-3} (\sqrt{(z^2 - a^2)(z^2 - b^2)} S(z) \pm iR(z)) \quad (42)$$

with some monic even polynomial S of degree 4; and an odd polynomial R with degree ≤ 5 .

The vanishing of the sum of the residues of ψ'/ψ on the torus shows that ψ has twelve zeroes. These have to be triple zeroes, since $\phi = \psi^{1/3}$ has no branching point and we conclude that

$$S^2(z)(z^2 - a^2)(z^2 - b^2) + R^2(z) = (z^2 - \lambda^2)^3(z^2 - \mu^2)^3 \quad (43)$$

This requirement allows us to solve the Potts model: for given α and g , we look for a, b and the coefficients of R, S (3 and 2 parameters); (43) gives 4 constraints (for a sixth-order polynomial (in z^2) to have two triple roots), and 3 other equations are given by the asymptotic expansion of f (coefficient of z ; eliminate c between the coefficients of z^0 and z^{-1} ; coefficient of z^{-2}).

²We thank B.Eynard for his advice

Precisely at the critical point, we know that $a = 0$ and $\rho(t_c + dt) \sim dt^{5/6}$ so that $\pi(u) \sim u^{5/3}$: inspection of 43 then shows that 0 is a zero of S with multiplicity 4, and with multiplicity 5 for R . We thus have identified all coefficients in 43

$$(z^4)^2 z^2 (z^2 - b^2) + (bz^5)^2 = (z^2)^3 (z^2)^3 \quad (44)$$

and we find that

$$\alpha_c^2 = \frac{\sqrt{47} - 3}{38}, g_c = 27\alpha_c^4 \sqrt{\frac{665}{486(\sqrt{47} - 3)}} \quad (45)$$

We then solve (43) in the vicinity of that point to investigate the scaling limit; to obtain a generic perturbation, we can shift g while keeping $\alpha = \alpha_c$. We find the following behaviours

$$db \sim a^2, \lambda \sim \mu \sim a^{5/6} \text{ with } \lambda^2 + \mu^2 \sim a^2 \quad (46)$$

the roots of R, S scale as a , while the leading coefficient of R varies with $a^{10/3}$, just like g . And $dc = a^2/2 + \dots$

For the scaling behaviour of f we get

$$f(z = a \cosh y + i0) = -\frac{i}{\alpha\sqrt{3}} \left[e^{-2i\pi/3} (ia^5 b \left[\sinh y (\cosh^4 y - \frac{3}{4} \cosh^2 y + 1/16) - \cosh y (\cosh^4 y - 5/4 \cosh^2 y + 5/16) \right])^{1/3} - e^{2i\pi/3} (ia^5 b \left[\sinh y (\dots) + \cosh y (\dots) \right])^{1/3} \right] \quad (47)$$

where the branch of the third root is fixed by continuation from the behaviour of f at infinity, and where the bracketed combinations of hyperbolic lines reduce remarkably to $-\exp(-5y/16)$ and $\exp(5y/16)$ so that

$$f(a \cosh 3y + i0) \sim a^{5/3} \cosh 5(y - 2i\pi/15) \quad (48)$$

The explicit limit we found for c allows us to see that ρ also has a scaling limit expressed with hyperbolic lines because $u \sim a \cosh 3y$ corresponds effectively to $dt \sim a^2 \cosh 6y$ (the coefficient $1/2$ in $dc = a^2/2 + \dots$ is really important here)

Finally, using $v_- \circ u_+ = id$ we find v_- , then g , the one-loop function for a Potts matrix (partition function of a disk with a uniform color along the boundary)

$$g(\epsilon^5 \cosh 5t) = \text{regular} + cst \epsilon^6 \cosh 6(t - i\pi/5) + \dots \quad (49)$$

Such scaling laws are universal among unitary models: in [8] however, a two-matrix formulation was used to produce any $(m, m+1)$ model, with multicritical points analogous to Kazakov's one matrix-model (the geometry of the different types of polygons used to build the surface -not only triangles- conspires with the Ising spins to produce different conformal models) and the identification with the explicit model here discussed is rather

unclear. We can just argue that in both cases, we compute averages of macroscopic insertions of the identity operator.

To conclude, we would like to consider again the 3-critical point: how could it appear in the first section, while we missed it in the above exact computation? To reach tricriticality, we need two coupling constants for the statistical model, and not only the temperature: it seems that we would obtain a model in the right universality class if we added an $\text{Tr}X^3$ term to the action, to produce a kind of dilute Potts gas; this modification would not change the lines of the first section, where the simplest type of criticality was assumed with given number of parameters, while it requires a new investigation of the derivation given in the last section. That seems interesting to clarify, in order to gain a better understanding of the $q \rightarrow 4$ limit, where these two RG fixed points merge into a first-order fixed point, this limit being the $c \rightarrow 1$.

References

- [1] V.A.Kazakov, Nucl. Phys. **B** (Proc. Suppl.) 4 (1988) 93
- [2] V.A.Kazakov, Mod.Phys.Lett. 4A (1989) 1691
- [3] C.Itzykson, J.-B.Zuber, J.Math.Phys. 21 (1980) 441
- [4] I.K.Kostov, Nucl. Phys. **B** (Proc.Suppl.) 10A (1989) 295
- [5] D.V.Boulatov, Mod. Phys. Lett. A8 (1993) 557
- [6] A.A.Migdal, Mod. Phys. Lett. A8 (1993) 359
- [7] E.Brézin, C.Itzykson, G.Parisi, J.-B.Zuber, Comm. Math. Phys. 59 (1978) 35
- [8] J.-M.Daul, V.A.Kazakov, I.K.Kostov, Nucl. Phys. **B409** (1993) 311
- [9] J.L.Cardý, in Phase Transitions and Critical Phenomena, Academic Press (Eds. C.Domb, J.L.Lebowitz) Vol. 11; B.Nienhuis, A.Berker, E.K.Riedel, M.Schick, Phys. Rev. Lett. 43 (1979) 737; B.Nienhuis, J.Phys.A 15 (1982) 199
- [10] E.Brézin, D.J.Gross, Phys. Lett. **B** 97 (1980) 120
- [11] D.J.Gross, M.J.Newman, Phys. Lett. **B** 266 (1991) 291